

Unit-4 (Recurrence relation)

⇒

Linear Recurrence Relation With Constant Coefficients

* Recurrence Relation:-

It is an equation that recursively defines a sequence where the next term is a function of the previous terms.

Ex

$$S = \{ 2, 2^2, 2^3, \dots, 2^n, \dots \}$$

$$S = \{ S_n \} = \{ 2^n \} \text{ Sequence } \{ n=1, 2, 3, \dots \}$$

Recurrence
relation

Initial Condition

$$S_n = 2 S_{n-1}$$

$$S_1 = 2$$

$$n=2$$

$$S_2 = 2 S_1 = 2(2) = 2^2 \quad n \geq 2$$

$$n=3$$

$$S_3 = 2 S_2 = 2(2^2) = 2^3$$

②

$$S = \{3, 7, 11, 15, 19, \dots\}$$

$$S_1 = 3$$

$$S_n = S_{n-1} + 4, \quad n \geq 2 \text{ is}$$

recurrence relation with
Initial Condition

③

$$S = \{1, 1, 2, 3, 5, 8, 13, \dots\}$$

$$S_n = S_{n-1} + S_{n-2}, \quad n \geq 3$$

↓ fibonacci sequence

with $S_1 = S_2 = 1$

* Linear recurrence relation
with Constant Coefficient

$$[C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n)]$$

where $C_0, C_1, C_2, \dots, C_k$ are
Constants.

Ex:-

① $2a_n + 3a_{n-1} = 3$ Linear

② $a_n - 7a_{n-1} + 12a_{n-2} = n \cdot 4^n$ Linear

③ $a_n = a_{n-1} + a_{n-2}$ Linear

④ $a_n = a_{n-1} \cdot a_{n-2}$ Not linear

⑤ $a_n \cdot a_{n-1} = a_{n-2}^2$ not-1.

Order :- It is the difference between the highest and lowest subscript of a

$n - (n-1) = 1$ Recurrence Relation

① $2a_n + 3a_{n-1} = 3$

② $a_n + 7a_{n-1} - 12a_{n-2} = 0$

③ $a_n + a_{n-1} + a_{n-2} + a_{n-3} = 0$

④ $a_n = 2a_{n-1} + 1$

order
0

1

2

3

1

* Degree :- It is defined as the highest power of a_n

* Degree of linear recurrence relation is 1.

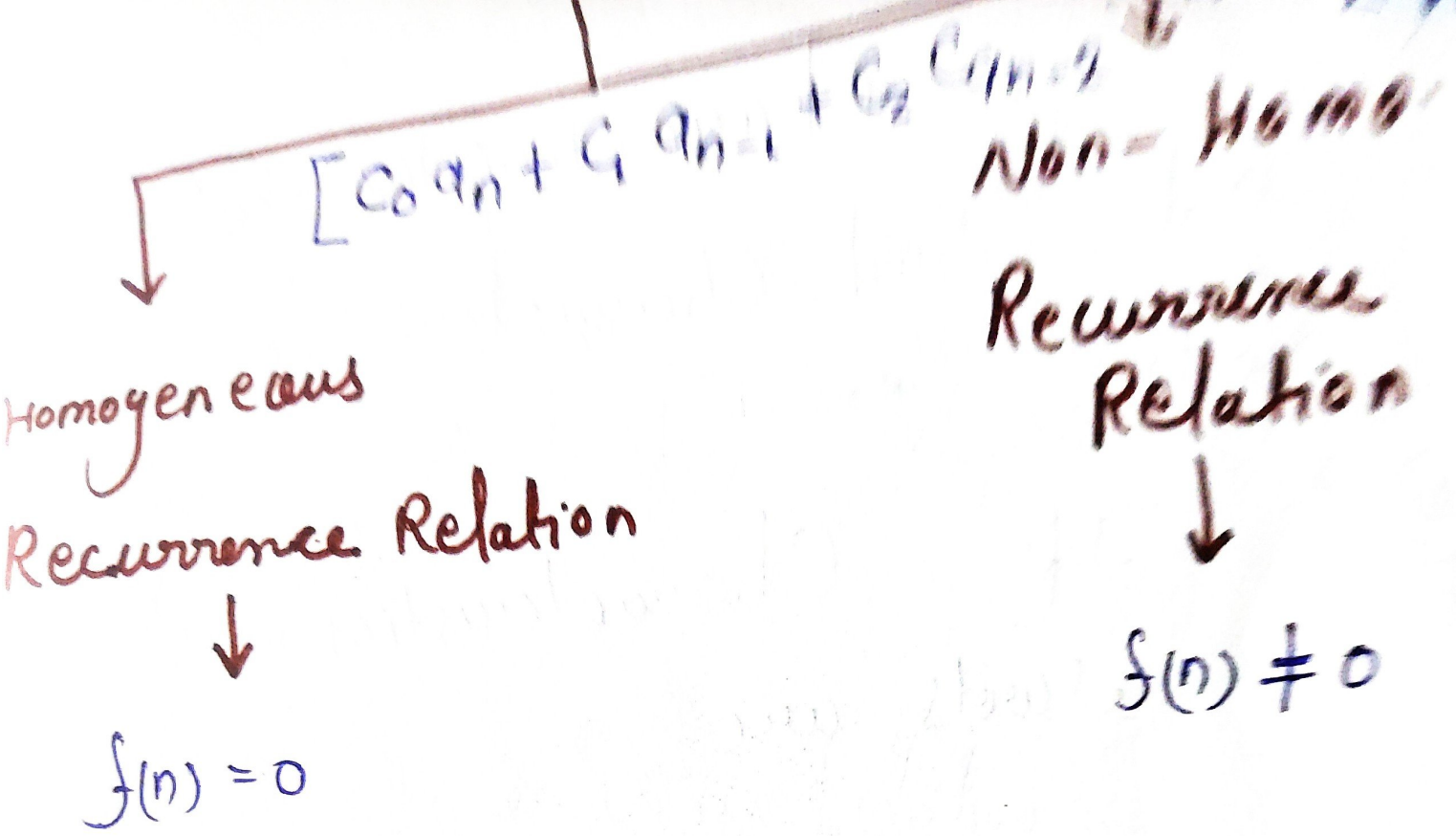
4^n

Methods of solving linear recurrence relation

- * Iteration Method
- * Method of characteristics roots
- * Generating functions.

The Method of Characteristic roots.

Linear Rec. Relation with
Constant Coefficient.



Homogeneous recurrence

Step to Solve

- ① put $a_n = x^n$, $a_{n-1} = x^{n-1}$ and so.
- ② Find an equation in terms of x . this is called as characteristic eqⁿ or Auxiliary equation.

$$[C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k}] = f(n)$$

Non-Homo.

Homogeneous

Recurrence Relation



$$f(n) = 0$$

Recurrence
Relation



$$f(n) \neq 0$$

Homogeneous recurrence

Step to solve

- ① put $a_n = x^n$, $a_{n-1} = x^{n-1}$ and so on
- ② Find an equation in terms of x . this is called as characteristic eqn or Auxiliary equation.

③ Solve characteristic eqⁿ
and find characteristic
roots

④ If characteristic
roots are

⑤ $\gamma = \gamma_1, \gamma_2, \gamma_3$ (distinct)

General Solⁿ is

$$a_n = b_1 \gamma_1^n + b_2 \gamma_2^n + b_3 \gamma_3^n$$

b_1, b_2, b_3 Constant

⑥ $\gamma = \gamma_1, \gamma_1$

$$a_n = (b_1 + n b_2) \gamma_1^n$$

$$(c) \quad \gamma = \gamma_1, \gamma_2, \gamma_3$$

$$\Rightarrow a_n = (b_1 + nb_2) \gamma_1^n + b_3 \gamma_2^n + b_4$$

$$(d) \quad \gamma = \underbrace{\gamma_1, \gamma_1, \gamma_1}_{\text{repeated}}, \gamma_2$$

$$a_n = (b_1 + nb_2 + n^2 b_3) \gamma_1^n + b_4$$

Q

X

Q/n

Solve $a_n = a_{n-1} + 2a_{n-2}$
; $n \geq 2$

With the initial Condition

$$a_0 = 0, a_1 = 1$$

$$a_n - a_{n-1} + 2a_{n-2} = 0$$

Put $a_n = r^n$

$$\Rightarrow r^n - r^{n-1} + 2r^{n-2} = 0$$

$$\Rightarrow r^n \left[1 - \frac{1}{r} - \frac{2}{r^2} \right] = 0$$

trivial

$$\Rightarrow \left(r^n \right) \left[\frac{r^2 - r - 2}{r^2} \right] = 0$$

$$\Rightarrow r^2 - r - 2 = 0$$

$$\boxed{r = 2, -1}$$

$$q_n = b_1(2)^n + b_2(-1)^n$$

Given $q_0 = 0, q_1 = 1$ $\rightarrow$ (1)

$$\Rightarrow q_0 = b_1 \cdot 2^0 + b_2(-1)^0$$

$$\textcircled{1} \quad b_1 + b_2 \rightarrow \textcircled{1}$$

$$\textcircled{2} \quad q_1 = 1$$

$$\Rightarrow q_1 = b_1 \cdot 2 + b_2(-1)$$

$$1 = 2b_1 - b_2 \quad \text{--- (ii)}$$

from (1) and (2)

~~$$\begin{aligned} b_1 + b_2 &= 0 \\ 2b_1 - b_2 &= 1 \\ \hline -b_1 + 2b_2 &= 1 \end{aligned}$$~~

$$\begin{aligned} (-1) \times b_1 + b_2 &= 0 \\ \times (1) \quad 2b_1 - b_2 &= 1 \end{aligned}$$

~~$$\begin{aligned} b_1 + b_2 &= 0 \\ 2b_1 - b_2 &= 1 \end{aligned}$$~~

$$-b_1 - b_2 = 0$$

$$2b_1 - b_2 = 1$$

$$-3b_1 = -1$$

$$b_2 = 1/3$$

$$b_1 = 1/3$$

$$\Rightarrow a_n = \frac{1}{3} 2^n + (-1/3) (-1)^n$$

Q2 Sol: $a_n = 4(a_{n-1} - a_{n-2})$

with Initial Condition.

$$a_0 = a_1 = 1$$

Soln

$$a_n - 4a_{n-1} + 4a_{n-2} = 0$$

Put $a_n = r^n$

$$\Rightarrow r^2 - 4r + 4 = 0$$

$$\underline{r^{n-2}}$$

$$\Rightarrow \cancel{r^2 - 4r + 4} \\ (r-2) \cdot (r-2) = 0$$

$$= \boxed{r = 2, 2}$$

$$\Rightarrow a_n = (b_1 + nb_2) 2^n \rightarrow (*)$$

condition $a_0 = a_1 = 1$

$$a_0 = 1$$

$$a_1 = 1$$

(i) $a_0 = b_1 2^{(0)}$

$$\boxed{1 = b_1}$$

(ii) $\boxed{a_1 = 1}$ in $(*)$

$$a_1 = (b_1 + b_2) 2$$

$$1 = (1 + b_2) 2 \Rightarrow 1 = 2 + 2b_2$$

$$= \boxed{\frac{-1}{2} = b_2}$$

Q1-2

$\Rightarrow$ put all in $(*)$

$$\boxed{a_n = \left(1 - \frac{n}{2}\right) 2^n}$$

Q3 Solve $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$

put $a_n = r^n$

$$r^3 - 8r^2 + 21r - 18 = 0$$

$$\Rightarrow r = 2, 3, 3$$

$$\begin{array}{r|rrrr} 1 & 1 & -8 & 21 & -18 \\ & & 2 & -12 & 18 \\ \hline & 1 & -6 & 9 & 0 \end{array}$$

$$\boxed{\lambda = 2}$$

$$1\lambda^2 - 6\lambda + 9 = 0$$

$$\lambda = 3, 3$$

Q4 $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$

with $a_0 = 8, a_1 = 6, a_2 = 26$

$$a_n + a_{n-1} - 4a_{n-2} - 4a_{n-3} = 0$$

put $a_n = r^n$

Soln

$$r^3 + r^2 - 4r - 4 = 0$$

$$\Rightarrow x^2(x+1) - 4(x+1) = 0$$

$$\Rightarrow (x^2+1)(x^2-4) = 0$$

$$\Rightarrow x = 1, 2, -2 \quad \begin{matrix} x^2 = +4 \\ \Rightarrow = 2 \end{matrix} \quad \begin{matrix} \sqrt{4} \\ \pm 2 \end{matrix}$$

$$\Rightarrow a_n = b_1 x^n + b_2 2^n + b_3 (-2)^n$$

(i) Condition

$$[a_0 = 8] \text{ put in } \textcircled{1}$$

$$a_0 = b_1 \cdot 1 + b_2 \cdot 2^0 + b_3 (-2)^0$$

$$[8 = b_1 + b_2 + b_3] \text{ --- (i)}$$

(ii) Condition

$$a_1 = 6$$

$$a_1 = b_1 + 2b_2 - 2b_3$$

$$[6 = b_1 + 2b_2 - 2b_3] \text{ --- (ii)}$$

(iii) Condition $[a_2 = 26]$

$$a_2 = b_1 + b_2 \cdot 2^2 + b_3 (-2)^2$$

$$\Rightarrow x^2(x+1) - 4(x+1) = 0$$

$$\Rightarrow (x^2+1)(x^2-4) = 0$$

$$\Rightarrow x = 1, 2, -2$$

$$x^2 = +4$$

$$x = \pm 2$$

$$\sqrt{4}$$

$$\pm 2$$

$$\Rightarrow a_n = b_1 x^n + b_2 2^n + b_3 (-2)^n$$

— (1)

condition

(i)

$$\boxed{a_0 = 8} \text{ put in (1)}$$

$$a_0 = b_1 \cdot 1 + b_2 2^0 + b_3 (-2)^0$$

$$\boxed{8 = b_1 + b_2 + b_3} \text{ — (i)}$$

condition

(ii)

$$a_1 = 6$$

$$a_1 = b_1 + 2b_2 + 2b_3$$

$$\boxed{6 = b_1 + 2b_2 + 2b_3} \text{ — (ii)}$$

(iii)

$$\text{condition } \boxed{a_2 = 26}$$

$$a_2 = b_1 + b_2 2^2 + b_3 (-2)^2$$

$$26 = b_1 + 4b_2 + 4b_3 \rightarrow \textcircled{3}$$

form ①, ②, and ③

$$\Rightarrow \boxed{b_1 = 2}, \boxed{b_2 = 5}, \boxed{b_3 = 1}$$

$$\Rightarrow \boxed{a_n = 2(-1)^n + 5(2)^n + 1(-2)^n}$$

Ans

Question for P.

Solve $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

With $a_0 = 1, a_1 = -2, a_2 = -1$?

Ans

$$a_n = (1 + 3n - 2n^2)(-1)^n$$

Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with

$$a_0 = 1 \text{ and } a_1 = 6 ?$$

As $a_n = (1+n)3^n.$

As

a_{n-3}

?

non-Homogeneous Recurrence Relation with Constant Coefficient

*
$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n),$$

 $f(n) \neq 0$

General Soln :-

$$a_n = \underbrace{a_n^{(h)}}_{\substack{\downarrow \\ \text{Homogeneous} \\ \text{Soln}}} + \underbrace{a_n^p}_{\substack{\downarrow \\ \text{Particular} \\ \text{Soln}}}$$

Case I

$$f(n) = \alpha \text{ (constant)}$$

- 1) let $a_n = A$
- 2) put $a_n = a_{n-1} = \dots = A$ in the given rec. relation

3. find A

4. $a_n^{(p)} = A$

Q1 Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$

with initial Condition $a_0 = 1$ and $a_1 = -1$

Solve

To find $a_n^{(h)}$

$$\Rightarrow a_{n+2} - 5a_{n+1} + 6a_n = 0$$

$$\Rightarrow \text{put } a_n = r^n$$

$$2, 3 = 6$$

$$\Rightarrow r^2 - 5r + 6 = 0$$

$$\Rightarrow r = 2, 3$$

$$a_n^h = b_1(3)^n + b_2(2)^n$$

To find a_n^p particular soln

$$a_{n+2} - 5a_{n+1} + 6a_n = 2$$

put $a_n = A$

$$\Rightarrow A - 5A + 6A = 2$$

$$\Rightarrow 2A = 2$$

$$\boxed{A = 1}$$

$$\Rightarrow \boxed{a_n^p = 1}$$

$$a_n \text{ g.s} = a_n^{(h)} + a_n^p$$

$$\Rightarrow \boxed{a_n = b_1 3^n + b_2 2^n + 1}$$

—★

(i) Condition

$$a_0 = 1$$

By *

$$a_0 = b_1 3^0 + b_2 2^0 + 1$$

$$1 = b_1 + b_2 + 1 \quad \text{--- (1)}$$

(ii) Condition

$$a_1 = -1$$

By *

$$a_1 = b_1 3^1 + b_2 2^1 + 1$$

$$-1 = 3b_1 + 2b_2 + 1$$

$$-2 = 3b_1 + 2b_2 \quad \text{--- (2)}$$

Solving (1) and (2)

$$\boxed{b_1 = -2}, \boxed{b_2 = 2}$$

$$\Rightarrow a_n = -2(3^n) + 2(2^n) + 1 \text{ Any}$$

Q 2

$$\text{Solve } a_n - 6a_{n-1} + 8a_{n-2} = 3$$

Solⁿ To find a^h

$$\Rightarrow a_n - 6a_{n-1} + 8a_{n-2} = 0$$

$$\text{put } a_n = x^n$$

$$\text{Char. eqⁿ is } x^2 - 6x + 8 = 0$$

$$\boxed{x = 2, 4}$$

$$\Rightarrow a_n^h = b_1(2)^n + b_2(4)^n$$

To find $a_n^{(p)}$

$$a_n - 6a_{n-1} + 8a_{n-2} = 3$$

put $q_n = A$

$$A - 6A + 8A = 3$$

$$\Rightarrow 3A = 3$$

$$\boxed{A = 1}$$

$$\therefore q_n^{(p)} = 1$$

$$\Rightarrow G.S = q_n^h + q_n^p$$

$$\boxed{q_n = b_1 2^n + b_2 4^n + 1}$$

Method of Ch roots for
Non-Homo. Linear Recurrence
Relation with Const Coeff.

$$[C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n)]$$

$f(n) \neq 0$

Case II

$f(n) = P(n)$, where $f(n)$ is a
polynomial of degree k .

Step to find a_n^P (Particular I)

- ① Let $a_n = A_0 + A_1 n + A_2 n^2 + \dots + A_k n^k$
- ② Put value of $a_n, a_{n-1}, \dots$ so on
in the given Eqⁿ
- ③ Compare the coeff of like powers

Method of Ch. Roots for
Non-Homo. Linear Recurrence
Relation with Const Coeff.

$$[C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n)]$$
$$f(n) \neq 0$$

Case II

$f(n) = P(n)$, where $f(n)$ is a
polynomial of degrees.

Step to find a_n^p (Particular I)

- ① Let $a_n = A_0 + A_1 n + A_2 n^2 + \dots + A_k n^k$
- ② Put value of $a_n, a_{n-1}, \dots$ so on
in the given Eqⁿ
- ③ Compare the Coeff of like power

4

Find $A_0, A_1, A_2, \dots, A_5$

5

$$Q_n^{(P)} = A_0 + A_1 n + \dots + A_5 n^5$$

(Putting values of A_i)

Ex: Solve

$$y_{n+2} - y_{n+1} - 2y_n = n^2$$

Soln To find y_n^h

degrees. put $y_n = r^n$

$$\Rightarrow r^2 - r - 2 = 0$$

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$$\boxed{r = 2, -1}$$

$-4A_5 n^5$

$$\boxed{y_n^h = b_1(-1)^n + b_2(2)^n}$$

so on

Perticular Integral

2 power on n

$$y_n^p = y_{n+2} - y_{n+1} - 2y_n = n^2$$

$$\text{put } y_n = A_0 + A_1 n + A_2 n^2$$

$$\Rightarrow \{A_0 + A_1(n+2) + A_2(n+2)^2\} -$$

$$\{A_0 + A_1(n+1) + A_2(n+1)^2\}$$

$$- 2\{A_0 + A_1 n + A_2 n^2\} = n^2$$

$$\Rightarrow \{A_0 + A_1(n+2) + A_2(n^2 + 4 + 4n)\}$$

$$- \{A_0 + A_1(n+1) + A_2(n^2 + 1 + 2n)\}$$

$$- 2\{A_0 + A_1 n + A_2 n^2\} = n^2$$

$$\Rightarrow \{A_0 + 2A_1 + 4A_2 - A_0 - A_1 - A_2 - 2A_0\}$$

$$+ \{A_1 + 4A_2 - A_1 - 2A_2 - 2A_1\}n$$

$$+ \{A_2 - A_2 - 2A_2\}n^2 = n^2$$

8) sec(1)

$$(-1)^n + \frac{11}{8} 3^n - 1 - \frac{n}{2}$$

Ans

~~with~~

$$-5a_{n+1} + 6a_n = n^2$$

$$b_1 \cdot 2^n + b_2 3^n + \frac{n}{2}$$

$$+ \frac{3}{2}n + \frac{5}{2}$$

$$a_n - 2a_{n-1} = 6n$$

$$= 10 \cdot 2^n - 6n - 12$$

$$\Rightarrow \{A_1 - 2A_0 + 3A_2\} + \{2A_2 - 2A_1\}n + \{-2A_2\}n^2 = n^2$$

Compare
~~Coeff~~ . Coeff.

$$A_1 - 2A_0 + 3A_2 = 0 \quad \text{--- (1)}$$

$$2A_2 - 2A_1 = 0 \quad \text{--- (2)}$$

$$-2A_2 = 1 \quad \text{--- (3)}$$

from 3, $A_2 = -1/2$

from (2) $A_1 = -1/2$

from (1) $A_0 = -1$

$$\Rightarrow y_n^{(p)} = A_0 + A_1 n + A_2 n^2$$

$$\boxed{y_n^{(p)} = -1 + (-1/2)n + (-1/2)n^2}$$

Ex 2 Solve $a_{n+2} + 2a_{n+1} - 15a_n = 16n + 10$

$a_0 = 1, a_1 = -1/2$

Soln To find $a_n^{(h)}$:-

$$a_{n+2} + 2a_{n+1} - 15a_n = 0$$

put $a_n = r^n$

Char. Eq :- $r^2 + 2r - 15 = 0$

$$\Rightarrow r = -5, 3$$

$$\Rightarrow a_n^{(h)} = b_1(-5)^n + b_2(3)^n$$

Particular Eqn $a_n^{(p)}$

$$\Rightarrow a_{n+2} + 2a_{n+1} - 15a_n = 6n + 10$$

$$\text{put } a_n = A_0 + A_1 n$$

$$\Rightarrow \{A_0 + A_1(n+2)\} + 2\{A_0 + A_1(n+1)\} - 15\{A_0 + A_1 n\} = 6n + 10$$

$$\Rightarrow \{A_0 + 2A_0 + 2A_1 + 2A_1 - 15A_0\} + \{A_1 + 2A_1 - 15A_1\}n = 6n + 10$$

$$\Rightarrow \{-12A_0 + 4A_1\} + (-12)A_1 n = 6n + 10$$

Comparing the coeff.

$$-12A_0 + 4A_1 = 10 \quad \text{--- (1)}$$

$$-12A_1 = 6 \quad \text{--- (2)}$$

from Eqⁿ (2)

$$A_1 = -1/2$$

from eq (1)

$$-12A_0 = 10 + 14 \times \frac{1}{2} = 12$$

$$A_0 = -1$$

$$\begin{aligned} Q_n^{(P)} &= A_0 + A_1 n \\ &= -1 - \frac{1}{2} n \end{aligned}$$

$$\therefore a_n = q_n^{(A)} + q_n^{(P)}$$

$$\Rightarrow a_n = b_1(-5)^n + b_2(3)^n - 1 - \frac{n}{2}$$

$$(i) a_1 = 1 \Rightarrow 1 = b_1 + b_2 - 1$$

$$\boxed{b_1 + b_2 = 2} \quad \text{--- (3)}$$

$$(ii) a_1 = -\frac{1}{2}$$

$$-\frac{1}{2} = -5b_1 + 3b_2 - \frac{3}{2}$$

$$\therefore -5b_1 + 3b_2 = 1 \quad \text{--- (4)}$$

Solving Eyn (3) and (4)

$$b_1 = 5/8$$

$$b_2 = 11/8$$

$$\Rightarrow \boxed{a_n = \frac{5}{8}(-5)^n + \frac{11}{8}3^n - 1 - \frac{n}{2}} \quad \text{Ans}$$

Question

Q1

$$a_{n+2} - 5a_{n+1} + 6a_n = n^2$$

$$\text{As } a_n = b_1 \cdot 2^n + b_2 3^n + \frac{n^3}{2} + \frac{3}{2}n + \frac{5}{2}$$

(2) Solve $a_n - 2a_{n-1} = 6n$

(i) $a_1 = 2$

Ans $a_n = 10 \cdot 2^n - 6n - 12$

Generating function

Ex 2

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Sequence $\rightarrow a_0, a_1, a_2, \dots, a_k, \dots$

$$G(x) = a_0x^0 + a_1x + a_2x^2 + \dots \\ \dots + a_kx^k + \dots$$

$$= \sum a_k x^k$$

↓
generating function

Ex 3

Ex 4

Ex:

1, 1, 1, ...

$$G(x) = 1x^0 + 1x + 1x^2 + 1x^3 + \dots$$

$$G(x) = 1 + x + x^2 + x^3 + \dots$$

Ex

Ex 6

Ex 2

2, 2, 2, ---

$$\begin{aligned} G(x) &= 2x^0 + 2x + 2x^2 + 2x^3 + \dots \\ &= 2 + 2x + 2x^2 + 2x^3 + \dots \end{aligned}$$

Ex 3

1, -1, 1, -1, 1, -1, ---

$$G(x) = 1 - x + x^2 - x^3 + x^4 + \dots$$

Ex 4

1, 2, 3, 4, 5, ---

$$G(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

Ex 5

0, 1, 2, 3, 4, ---

$$G(x) = 0 + x + 2x^2 + 3x^3 + 4x^4 + \dots$$

Ex 6

1, a, a^2, a^3, ---

$$G(x) = 1 + ax + a^2x^2 + a^3x^3 + \dots$$

Ex 7

$$1, -a, a^2, -a^3, \dots$$

$$G(x) = 1 - ax + a^2x^2 - a^3x^3 + \dots$$

* Closed form of generating function *

Q1 $a_n = 1, 1, 1, \dots$ or $a_n = 1$

$$\Rightarrow G(x) = 1 + x + x^2 + x^3 + \dots$$

$$\Rightarrow \boxed{G(x) = \frac{1}{1-x}}$$

$$|x| < 1$$

U.P.
 $\frac{a}{1-x}$

Ex 2

$$2, 2, 2, 2, \dots$$

$$a_n = 2$$

Ex 7 $1, -a, a^2, -a^3, \dots$

$$G(x) = 1 - ax + a^2x^2 - a^3x^3 + \dots$$

* Closed form of generating function *

Q1 $a_n := 1, 1, 1, \dots$ or $a_k = 1$

$$\Rightarrow G(x) = 1 + x + x^2 + x^3 + \dots$$

$$\Rightarrow \boxed{G(x) = \frac{1}{1-x}}$$

$$|x| < 1$$

U.P.
 $\frac{a}{1-x}$

Ex 2

$2, 2, 2, 2, \dots$

$a_k = 2$

$$\Rightarrow G(x) = 2 + 2x + 2x^2 + 2x^3 + \dots$$

$$\Rightarrow G(x) = 2 [1 + x + x^2 + x^3 + \dots]$$

$$\Rightarrow G(x) = \frac{2}{1-x}$$

10th

G.P

$$a + ar + ar^2 + \dots$$

$$\frac{a}{1-r}$$

Ex:- 3 1, 2, 3, 4, 5, \dots

$$a_k = k+1$$

$$G(x) = 1 + 2x + 3x^2 + 4x^3 + \dots \quad \text{--- (1)}$$

multiply(x)

$$x \cdot G(x) = x + 2x^2 + 3x^3 + 4x^4 + \dots \quad \text{--- (2)}$$

$$\text{(1) - (2)}$$

$$(1-x) G(x) = 1 + x + x^2 + x^3 + \dots$$

$$(1-x) G(x) = \frac{1}{1-x}$$

$$G(x) = \frac{1}{(1-x)^2}$$

⑤

0, 1, 2, 3,

$$G(x) = 0 + 1x + 2x^2 + 3x^3 + \dots$$

$$G(x) = x + 2x^2 + 3x^3 + \dots$$

$$= x [1 + 2x + 3x^2 + \dots]$$

$$= x \left[\frac{1}{(1-x)^2} \right]$$

$$G(x) = \frac{x}{(1-x)^2}$$

Example 3

⑧

⑥

0, 0, 1, 1, 1,

$$G(x) = 0 + 0x + 1x^2 + 1x^3 + 1x^4 + \dots$$

$$= x^2 [1 + x + x^2 + \dots]$$

$$G(x) = \frac{x^2}{1-x}$$

Example 1

⑦

1, 1, 0, 1, 1, 1, ...

$$G(x) = 1 + x + \underbrace{0x^2}_{\text{add and subtract } (x^2)} + 1x^3 + x^4 + \dots$$

$$G(x) = (1 + x + x^2 + x^3 + \dots) - x^2$$

$$G(x) = \frac{1}{1-x} - x^2$$

⑧

1, 0, -1, 0, 1, 0, -1, ...

~~$$G(x) = 1 + 0x + 0x^2 + \dots$$~~

$$G(x) = 1 + 0x - x^2 + 0x^3 + 1x^4 + 0x^5 - x^6 + \dots$$

$$G(x) = 1 - x^2 + x^4 - x^6 + \dots$$

$$G(x) = 1 + (-x^2) + x^4 + (-x^6) + \dots$$

$$G(x) = \frac{1}{1+(x^2)}$$

Direct formula

SNo	a_k	$G(x)$
1.	1	$\frac{1}{1-x}$
2.	$(-1)^k$	$\frac{1}{1+x}$
3.	C Constant	$\frac{C}{1-x}$
4.	$k+1$	$\frac{1}{(1-x)^2}$
5.	k ($k = 0, 1, 2, \dots$)	$\frac{x}{(1-x)^2}$
6	a^k	$\frac{1}{1-ax}$
7	$(-a)^k$	$\frac{1}{1+ax}$
8.	ka^k	$\frac{ax}{1-ax^2}$
9	k^2	$\frac{x(1+x)}{(1-x)^3}$

Q Find the closed form of generating functions

(a) $a_k = 3$

$$G(x) = \frac{3}{1-x} \quad | \text{ By formula No. 3}$$

(b) $a_k = k+3$

$$= \frac{x}{(1-x)^2} \quad | \text{ By formula No. 5}$$

$$+ \frac{3}{1-x} \quad | \text{ By formula No. 3}$$

~~(c) $G(x) = \frac{x}{(1-x)^2}$ ✓~~

(c) $a_k = 3^k$

$$= \frac{1}{1-3x} \quad | \text{ By 6}$$

(d)

$$a_k = k(3+5k)$$

$$a_k = 3k + 5k^2$$

$$\Rightarrow G(x) = \frac{3x}{(1-x)^2} + \frac{5x(1+x)}{(1-x)^3}$$

$\downarrow$ By .5 $\downarrow$ By .9

Q Find the Closed form of the generating function

(a) $0, 0, 0, 1, 1, 1, \dots$

(b) $3, -3, 3, -3, 3, -3, \dots$

(c) $3, 9, 27, 81, \dots$

(a) Soln

$$\begin{aligned}
 G(x) &= 0 + 0x + 0x^2 + x^3 + x^4 + \dots \\
 &= x^3 [1 + x + x^2 + \dots] \\
 &= \frac{x^3}{1-x}
 \end{aligned}$$

(b) Soln
 $G(x) = 3x - 3x^2 + 3x^3 - 3x^4 + \dots$

$$G(x) = 3 \cdot [1 - x + x^2 - x^3 + \dots]$$

$$G(x) = \frac{3}{1+x}$$

(c) Soln

$$G(x) = 3 + 9x + 27x^2 + 81x^3 + \dots$$

$$G(x) = 3 [1 + 3x + 3^2x^2 + 3^3x^3 + \dots]$$

$$G(x) = \frac{3}{1-3x}$$
